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EQUATIONS OF HYDRODYNAMICS FOR POROUS MEDIA WITH A VOID STRUCTURE POSSESSING FRACTAL GEOMETRY

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We obtain the equations of filtration for the case when a void space is a fractal embedded in a continuous medium. We consider a model of capillary permeation of porous materials with percolation properties.

Introduction. In recent years the theory of fractals, i.e., objects with a fractional spatial dimensionality [1-4], has been widely used to describe the structure of disordered media and processes in disordered media. Examples of disordered materials are porous bodies such as rocks. The void space, the skeleton, or the surface of the body can be a fractal [5, 6].

It was shown in [7, 8] that certain features of the behavior of processes in porous media are determined by the percolation properties of the void space of rocks. It is known (see [9], for example) that a percolation cluster has fractal properties. Hence it follows that the void space of sedimentary rocks has fractal properties, which is observed experimentally [10].

1. Multiphase Filtration. We first consider the equations of multiphase filtration for the case when one phase (the wetting phase, for example) extrudes the other (nonwetting) phase.

We assume that the void space is a fractal with the Hausdorff−Besikovitch dimensionality d_f embedded in a continuous medium with dimensionality d ($d \geq d_f$, d = 2, 3).

To derive the equations we adopt the method used in [11] to derive the equations of filtration in a fractured medium with the cracks having a fractal geometry.

We consider flow with cylindrical (d = 2) or spherical (d = 3) symmetry, when all functions depend only on the time t and the distance r from the center of symmetry. Then the integral conservation of mass equation is written in the form

$$\frac{\partial}{\partial t} \int_{r_0 \leqslant r \leqslant r_1} m\rho_w S_w d^{d_f}\mu_p = \int_{r=r_0} q_f d^{d_f}\mu_s - \int_{r=r_1} q_f d^{d_f}\mu_s, \tag{1}$$

where $d^{d_f}\mu_p = dr d^{d_f}\mu_s$.

Below we will need the relation

$$\int_{r=r_1} d^{d_f}\mu_s = r_1^{d_f-1} \int_{r=1} d^{d_f}\mu_s = r_1^{d_f-1}\alpha_{d_f}, \tag{2}$$

where $\alpha_{d_f} = 2\pi^{d_f/2}\Gamma^{-1}(d_f/2)$ is the surface area of a unit ($d_f - 1$)-dimensional sphere.

Using (2), we rewrite (1) in the form

$$m\rho_w \frac{\partial S_w}{\partial t} = \frac{1}{r^{d_f-1}} \frac{\partial}{\partial r} [r^{d_f-1}q]. \tag{3}$$

Combining (3) with the equation of the generalized Darcy's law

$$\mathbf{u}_i = -\frac{k}{\mu_i} f_i(S_w)\nabla p_i, \tag{4}$$

and with the following expression for the pressure difference in the phases

$$p_2 - p_1 = p_0(S_w) \qquad (5)$$

and using the condition that the total filtration velocity is constant

$$u_1 + u_2 = u(t) = \text{const}, \qquad (6)$$

we obtain the Rappoport—Leese equation for a porous medium with fractal properties

$$m \frac{\partial S_w}{\partial t} + \frac{u}{r^{d_f-1}} \frac{\partial}{\partial r} [r^{d_f-1} F(S_w)] = \frac{k}{\mu_2} \frac{1}{r^{d_f-1}} \frac{\partial}{\partial r} \left[r^{d_f-1} \Phi(S_w) \frac{\partial S_w}{\partial r} \right], \qquad (7)$$

$$\Phi(S_w) = -\frac{f_1(S_w) f_2(S_w)}{f_1(S_w) + \mu_0 f_2(S_w)} p_0'(S_w) = f_2(S_w) F(S_w) p_0'(S_w);$$

$F(S_w) = f_1(S_w)/[f_1(S_w) + \mu_0 f_2(S_w)]$ is the flux distribution function and $\mu_0 = \mu_1/\mu_2$ is the ratio of the viscosities of the wetting (water) and nonwetting (oil) phases.

<u>2. Dynamics of Impurities in a Homogeneous Liquid</u>. The total impurity flux q in the porous medium is composed of the convection and diffusion fluxes [12]:

$$q = c\rho_w u - D_u \frac{\partial c}{\partial r}, \qquad (8)$$

where c is the concentration of impurities in the aqueous phase, D_u is the convective diffusion coefficient, and u is the velocity of convective flow.

Using (1)-(3) and (8), we obtain the equation of motion of the impurities in a homogeneous liquid ($\rho_w = 1$)

$$\frac{\partial c}{\partial t} + \frac{u}{m} \frac{1}{r^{d_f-1}} \frac{\partial}{\partial r} [r^{d_f-1} c] = \frac{D_u}{m} \frac{1}{r^{d_f-1}} \frac{\partial}{\partial r} \left[r^{d_f-1} \frac{\partial c}{\partial r} \right]. \qquad (9)$$

Phenomena associated with adsorption can be taken into account in (9) without difficulty.

<u>3. Model of Capillary Permeation of Porous Materials with Percolation Properties</u>. The effect of capillary forces on extrusion is crucial in many situations of interest in the petroleum industry (inhomogeneous and fractured strata).

The process of backflow capillary permeation is described by (7) with the total flow velocity u equal to zero:

$$m \frac{\partial S_{w'}}{\partial t} = \frac{k}{\mu_2} \frac{1}{r^{d_f-1}} \frac{\partial}{\partial r} \left[r^{d_f-1} \Phi(S_w) \frac{\partial S_w}{\partial r} \right]. \qquad (10)$$

This is a quasilinear equation of the parabolic type. Equations of this type occur in mathematical models of diverse phenomena and processes in mechanics, physics, technology, biophysics, biology, ecology, and many other fields [13].

It is known [14] that equations of the type (10) where the function $\Phi(S_s)$ decreases and approaches zero as $S_w \to 0$ describe a process with a finite velocity of propagation of disturbances (at each instant of time the disturbance will extend out only over a finite region of space).

The initial and boundary conditions for (10) are

$$S_w(r, 0) = S_{wres}; S_w(0, t) = S_{wres}^*; S_w(r_f, t) = S_{wres}, \qquad (11)$$

where S_w^* is the maximum value of the saturation of the porous sample by water and r_f is the coordinate of the permeation front.

In recent years percolation ideas [15-18] have often been used to explain processes in porous media when one liquid is extruded by another. In particular, percolation ideas make it possible to explain in a consistent way the typical features of processes in porous media observed experimentally. It was shown in [7, 8] that certain features of the behavior of processes in rocks are determined by the percolation properties of the void space of the rock.

The process of extrusion of one phase by another can be represented as the formation of an infinite cluster of saturated (by the wetting or nonwetting phase) pores. It is necessary to distinguish two situations:

1) the saturation S_w of the porous medium by the wetting phase is large (or the saturation S_p by the nonwetting phase is small). Then we have the formation of an infinite cluster of pores saturated by the nonwetting phase (an "oil" infinite cluster) [16];

2) when S_w is small (or S_n is large) we have the formation of a "water" infinite cluster.

We first consider the formation of an oil infinite cluster. It is known that the problem of extrusion by mercury introduced into the sample and extrusion of water by air in the semipermeable membrane method can be formulated in terms of percolation theory. In this case initially the filtration of the nonwetting phase will be equivalent to the formation of an infinite cluster of filtering pores in the sample [16, 19]. An infinite cluster exists when $p_0 > p_{0c}$, and the volume fraction of the nonwetting phase near the percolation threshold, i.e., for small $\Delta p_0 = p_0 - p_{0c} > 0$ will have the asymptotic behavior

$$S_p \sim (\Delta p_0)^\beta, \tag{12}$$

where β is the critical index for the probability that a filled pore belongs to the infinite cluster; p_{0c} is the breakdown pressure (the percolation threshold).

Because of the percolation properties of the void space of the porous medium one can use the scaling invariance hypothesis, according to which the geometry of the percolation system, including the structure of the infinite cluster, is determined by a single parameter: the correlation length L [20]. When $p_0 \to p_{0c}$ the correlation length diverges according to the power law

$$L \sim (p_0 - p_{0c})^{-\nu}, \tag{13}$$

where ν is the critical index for the correlation length.

The existence of a correlation length in the system implies that the saturation can be written as a function of distance [21]:

$$S(r) = S(p_0) f [r/L(p_0)], \tag{14}$$

where the dimensionless function $f(r/L) \to 1$ when $r/L \to \infty$.

Near the breakdown point (corresponding to the percolation threshold) the correlation length L is large and there exists a region of r satisfying the inequality $l \ll r \ll L$ in which the saturation should not depend on the small quantity $(p_0 - p_{0c})$ (here l is the minimum dimension in the system).

Because $S_p \sim (p_0 - p_{0c})^\beta$ and $L \sim (p_0 - p_{0c})^{-\nu}$ when $r/L \ll 1$, we must have $f(r/L) \sim (r/L)^{-\beta/\nu}$. Hence over distances less than the correlation length the saturation falls off with distance as the power law

$$S_p(r) \sim r^{-\beta/\nu} \quad \text{for} \quad l \ll r \ll L. \tag{15}$$

For small S_p the function $\Phi(S_p)$ can be approximated by the power law

$$\Phi(S_p) = A \left(\frac{S_p - S_{pres}}{S_p^* - S_{pres}} \right)^n, \tag{16}$$

where A is a constant of order unity, S_{pres} is the residual oil saturation, and S_p^* is the maximum possible value of the saturation by the nonwetting phase.

Noting that S_p in (15) can be replaced by $(S_p - S_{pres})/(S_p^* - S_{pres})$ and substituting it into (16), we obtain Φ as a function of r

$$\Phi(r) \sim r^{-\theta}, \tag{17}$$

where $\theta = n\beta/\nu$.

Equation (10) for the nonwetting phase can be written in the form

$$\frac{\partial S_p}{\partial t} = \frac{D}{r^{d_f - 1}} \frac{\partial}{\partial r} \left[r^{d_f - 1 - \theta} \frac{\partial S_p}{\partial r} \right], \tag{18}$$

where $d \sim k/(\mu_2 m)$.

For the second case, where the saturation by the wetting phase S_w is small, all of the preceding discussion remains in force if we introduce the new variables $S_w = 1 - S_p$ (the probability that a bond belongs to the aqueous infinite cluster) and $p = 1 - p_0$ (the probability that a bond belonging to pores filled by the wetting phase is conducting). In this case $\Phi(S_w)$ can be approximated as

$$\Phi(S_w) = A_1 \left(\frac{S_w - S_{wres}}{S_w^* - S_{wres}} \right)^{n_1}.$$

Repeating the above discussion, we obtain a differential equation for S_w

$$\frac{\partial \overline{S}_w}{\partial t} = \frac{D_1}{r^{d_f-1}} \frac{\partial}{\partial r}\left[r^{d_f-1-\theta_1} \frac{\partial \overline{S}_w}{\partial r}\right], \tag{19}$$

where $\theta_{1'} = n_1\beta_1/\nu_1$, $\overline{S}_w = (S_w - S_{wres})/(S_w{}^* - S_{wres})$, S_{wres} is the residual saturation by the wetting phase and $S_w{}^*$ is the maximum possible value of the saturation by the wetting phase.

Equations of the type (18), (19) with diffusion coefficients dependent upon distance have been encountered earlier in the study of diffusion in a fractal, in a percolation cluster [22, 9], and in turbulent media [23, 24]. The source function, the mean-square displacement, and the probability that a particle is at the initial lattice point were found in [22, 23].

We determine the solution of (19). We introduce the self-similar variable

$$\eta = \frac{r^{2+\theta_1}}{(2+\theta_1)^2 D_1 t} \tag{20}$$

and assume a solution in the self-similar form $\overline{S}_w = f(\eta)$ with the following initial and boundary conditions:

$$\overline{S}_w(\eta,\, 0) = 0; \quad \overline{S}_w(0,\, t) = 1; \quad \overline{S}_w(1,\, t) = 0. \tag{21}$$

In terms of the new variables (19) becomes

$$\frac{\partial^2 \overline{S}_w}{\partial \eta^2} + \left(1 + \frac{d_f}{(2+\theta_1)\eta}\right) \frac{\partial \overline{S}_w}{\partial \eta} = 0. \tag{22}$$

Let $\xi = \partial \overline{S}_w/\partial \eta$. Then (22) takes the form

$$\frac{\partial \xi}{\partial \eta} + \left(1 + \frac{d_f}{(2+\theta_1)\eta}\right) \xi = 0. \tag{23}$$

The solution of (23) is

$$\xi = \frac{\partial \overline{S}_w}{\partial \eta} = C_1 \exp\left(-\eta\right) \eta^{-d_f/(2+\theta_1)}. \tag{24}$$

Integrating (24), we obtain

$$\overline{S}_w = C_1 \int\limits_{\eta}^{1} \exp\left(-\eta\right) \eta^{-d_f/(2+\theta_1)} d\eta + C_2. \tag{25}$$

We find from the boundary conditions (21)

$$C_2 = 0, \quad C_1 = \left[\int\limits_{0}^{1} \exp\left(-\eta\right) \eta^{-d_f/(2+\theta_1)} d\eta\right]^{-1} = [\gamma\left(1 - d_f/(2+\theta_1),\ 1\right)]^{-1},$$

where $\gamma(a,\, x) = \int\limits_{\varrho}^{x} \exp(-t) t^{a-1} dt$ is the incomplete gamma function [25].

The final solution can be written as

$$S_w = S_{wres} + \frac{S_w{}^* - S_{wres}}{\gamma\left(1 - d_f/(2+\theta_1),\ 1\right)} \int\limits_{\eta}^{1} \exp\left(-\eta\right) \eta^{-d_f/(2+\theta_1)} d\eta. \tag{26}$$

The initial and boundary conditions for (18) will be

$$\overline{S}_p(\eta,\, 0) = 1; \quad \overline{S}_p(0,\, t) = 0; \quad \overline{S}_p(1,\, t) = 1, \tag{27}$$

where

$$\overline{S}_p = \frac{S_p - S_{pres}}{S_p{}^* - S_{pres}}.$$

Then the final expression for the solution is

$$S_p = S_{pres} + \frac{S_p{}^* - S_{pres}}{\gamma\left(1 - d_f/(2+\theta),\ 1\right)} \int\limits_{0}^{\eta} \exp\left(-\eta\right) \eta^{-d_f/(2+\theta)} d\eta. \tag{28}$$

The solution (26), (28) generalizes the expressions obtained earlier in [26] for the model of capillary permeation in the one-dimensional case.

CONCLUSIONS

1. We have obtained the equations of multiphase filtration and motion of the impurities in a homogeneous liquid for the case when the void space is a fractal with the Hausdorff—Besikovitch dimensionality d_f embedded in a continuous medium with dimensionality d.

2. We have presented a model of capillary permeation of porous materials with percolation properties and its solution.

NOTATION

d_f, Hausdorff – Besikovitch dimensionality; S_w, S_n, saturations of the porous material by the wetting and nonwetting phases; ρ_w, density of water; q_f, radial component of the flux; $d\mu_н^{d_f}$, Hausdorff measure of a space with dimensionality d_f; $d\mu_s^{d_f}$, Hausdorff dimensionality of a cross section of the fractal by a sphere of radius r; $f_1(S_w)$, $f_2(S_w)$, relative phase permeabilities; $p_0(S_w)$, capillary pressure; m, porosity of the sample; k, absolute permeability of the sample; β, ν, critical indices; L, correlation length.

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